Explicit Guidance Along an Optimal Space Curve

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An explicit guidance theory is developed for maneuvering to a prescribed destination with terminal constraints on velocity vector direction. Motion is constrained to an *optimal*, three-dimensional space curve by constraint forces perpendicular to velocity (lift). Lift components are derived by twice differentiating functions specifying radial distance and geocentric latitude as functions of longitude. An optimal space curve is determined by solving a two-point boundary-value problem in the calculus of variations. Necessary conditions for an extremum are 1) a set of coupled, fourth-order, Euler–Lagrange differential equations for the space curve functions; 2) a single, first-order differential equation for the adjoint variable; and 3) boundary conditions specified at two ends of the trajectory. Although energy is not conserved because of drag, motion along the space curve is integrable because lift-induced drag is determined by trajectory curvature. Velocity along the space curve may be expressed by a quadrature evaluated by the method of successive approximation to refine the accuracy of the compressibility drag slowdown.

Background

FUTURE maneuvering vehicles, such as the space plane, will require advanced midcourse guidance algorithms to optimize performance and arrive at a prescribed destination with terminal constraints on flight path. During flight in the atmosphere, vehicle orientation relative to the velocity vector (angle of attack) is controlled to generate the required acceleration perpendicular to velocity (lift). Many explicit guidance algorithms have been developed for lift-controlled entry vehicles.[1–21] The guided trajectory problem is not generally integrable, except in certain cases (discussed shortly).

Integrable cases for lifting trajectories include constant lift-to-drag L/D ratio, constant-bank angle, and equilibrium glide at constant flight-path angle. Hodograph space solutions express velocity magnitude by a function of turning angle, and the configuration space trajectory is determined by a quadrature.[4–8] These approximate solutions are useful for preliminary design of midcourse trajectories satisfying mission performance objectives within vehicle aerothermodynamic limitations (trim, loads, and heating).

Direct methods have been used extensively to develop explicit, optimal guidance algorithms. For a vehicle with bounded lift control, optimal range extension maneuvers consist of maximum and minimum L/D subarcs connected by intermediate cruise segments.[9–11] Numerical solutions may be ill-conditioned because switching points must be determined to satisfy the boundary conditions. Green's Theorem may be applied to determine the sequence of maximum and minimum L/D subarcs (without cruise segments) that optimize performance while satisfying end conditions on altitude and flight path.[12,13]

Optimal, modulated-lift trajectories admit approximate analytic solutions characterized by a change of independent variable from time to an appropriate trajectory variable. For example, proportional navigation guidance minimizes control effort or time-integrated lift acceleration magnitude, and closed-form trajectory solutions are obtained when the new independent variable is line-of-sight angle.[14–16] For maximum velocity turns to a specified heading, approximate optimal control histories were derived from an integrable system of equations using flight-path angle[17] or range[18] as new independent variables. Important complications can arise when the new independent variable does not vary monotonically; for example, depending on boundary conditions, when inflection points or heading reversals occur.

Optimal perturbation guidance algorithms minimize performance sensitivity to off-nominal initial conditions or in-flight deviations from a precomputed optimal trajectory. Selection of weighing matrices is an important issue in the design of optimal regulators using linear-quadratic synthesis techniques.[19–21] It has been shown that controller robustness improves when 1) trajectory variables (e.g., altitude) are used, either as independent variable (rather than time) or in the performance index; and 2) uncontrollable states (e.g., velocity) are deweighted.

Approach

In this article, inverse methods are used to develop an explicit guidance theory for unthrusted, modulated-lift entry vehicles maneuvering to prescribed destination with terminal constraints on velocity vector direction. Geometrical trajectory shape is specified by expressing altitude and geocentric latitude as functions of Earth-fixed longitude. Two differentiations of each trajectory function determine slopes and curvatures, thus specifying velocity-path angles and lift-acceleration components. An optimal space curve can be determined by solving a two-point boundary-value problem in the calculus of variations, in which only one state variable (velocity along the curve) and adjoint variable are needed.

Inverse methods are commonly used in celestial mechanics to prescribe *exoatmospheric* orbits. Conservative forces are expressed by the gradient of a potential function U determined by solving a partial differential equation in the planar case,[22,23] or a system of partial differential equations in the three-dimensional case.[24] When U does not involve time explicitly, energy is conserved along an individual orbit, and velocity is a known function of position along the orbit.

For an unthrusted vehicle in endoatmospheric flight, energy is not conserved since velocity decreases monotonically because of drag. Although not directly controllable using lift, the velocity profile depends on trajectory shape through time-of-flight and lift-induced drag. Time-of-flight can be modulated by flight-path angle, while trajectory curvature determines lift magnitude and lift-induced drag. With the space curve approach, it will be shown that velocity magnitude may be expressed by a quadrature evaluated by the method of successive approximation, which was used previously for ballistic trajectories decelerating through transonic velocities.[25–27]

Inverse methods have been used to synthesize nonlinear autopilots[28] and guidance algorithms. In the former case,

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attitude control torques may be derived from a prescribed angle-of-attack history during *transient* response, but steady-state attitude motions are specified by lift, or angle-of-attack, requirements defined by the guidance algorithm. This article extends the *characteristic curve* guidance algorithm[1,2] by 1) developing a space curve optimization theory, and 2) demonstrating the integrability of motion along the curve.

Formulation

A minimum of six dynamical degrees of freedom are required for the closed-loop guidance and control of a maneuvering entry vehicle. Time-varying lift commands can be executed almost instantaneously because body angular rates are generally much faster than flight-path angular rates. Although control accelerations are not always negligible, an acceleration command autopilot will adjust the controls to obtain the desired acceleration profile. Therefore, trajectory and attitude motions may be uncoupled, reducing the original problem to two, uncoupled, three-degree-of-freedom problems. Although beyond the scope of this article, attitude control policy must be carefully matched with guidance requirements and vehicle aerodynamic configuration to achieve control-efficient maneuvers.[2]

Vehicle position and velocity are resolved in a topocentric frame $\mathscr{T}$ defined by the unit vectors $\mathbf{r}$ (radial), $\mathbf{s}$ (east), and $\mathbf{n}$ (north):

$$\mathbf{r} = \frac{\boldsymbol{r}}{r}, \qquad \mathbf{s} = \frac{\mathbf{K} \times \mathbf{r}}{|\mathbf{K} \times \mathbf{r}|}, \qquad \mathbf{n} = \mathbf{r} \times \mathbf{s}$$

where $\boldsymbol{r}$ is the geocentric position vector. Geocentric longitude λ and latitude θ angles are the spherical coordinates of $\mathbf{r}$ relative to an Earth-fixed, rotating frame with unit vector $\mathbf{K}$ along the Earth's polar axis (Fig. 1). The inertial angular velocity $\boldsymbol{\Omega}^*$ of the $\mathscr{T}$ frame is

$$\Omega^* = \omega_\oplus + \Omega, \qquad \omega_\oplus = \omega_\oplus \mathbf{K}, \qquad \Omega = \dot{\lambda}\mathbf{K} - \dot{\theta}\mathbf{s}$$

$\boldsymbol{\omega}_\oplus$ is the Earth's angular velocity with respect to inertial space. $\boldsymbol{\Omega}$ is the $\mathscr{T}$-frame angular velocity relative to Earth, caused by vehicle translation.

Three translational degrees of freedom describe lifting motion of a constant mass vehicle in a gravitational field. The equations of motion may be written as[11,27]

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + \boldsymbol{\Omega} \times \boldsymbol{u} = \boldsymbol{D} + \boldsymbol{L} - \frac{\mu_\oplus}{r^3}\boldsymbol{r} + \boldsymbol{a} \qquad (1a)$$

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} + \boldsymbol{\Omega} \times \boldsymbol{r} = \boldsymbol{u} \qquad (1b)$$

where the time derivative is taken relative to the $\mathscr{T}$ frame. Aerodynamic lift $\boldsymbol{L}$ and drag $\boldsymbol{D}$ accelerations depend on velocity $\boldsymbol{u}$ relative to a quiescent (wind-free) atmosphere rotating with the Earth:

$$\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{\omega}_\oplus \times \boldsymbol{r}$$

where $\boldsymbol{v}$ is the inertial velocity. Atmospheric rotation causes a misalignment angle between $\boldsymbol{u}$ and $\boldsymbol{v}$, which should be included to accurately resolve $\boldsymbol{L}$ and $\boldsymbol{D}$, since these accelerations can be significant.

Aerodynamic lift and drag accelerations uncouple in a velocity reference frame defined by the unit vectors

$$\mathbf{U} = \frac{\boldsymbol{u}}{u}, \qquad \mathbf{N} = \frac{\mathbf{r} \times \mathbf{U}}{|\mathbf{r} \times \mathbf{U}|}, \qquad \mathbf{P} = \mathbf{U} \times \mathbf{N}$$

Lift and drag may be expressed by

$$\boldsymbol{L} = L_h\mathbf{N} + L_v\mathbf{P}, \qquad \boldsymbol{D} = -D\mathbf{U}$$

since $\mathbf{N}$ and $\mathbf{P}$ are perpendicular to $\mathbf{U}$. For instantaneous vehicle response, it will be shown that lift components L_h and L_v depend on trajectory curvature. Drag-deceleration magnitude is expressed by

$$D = -\tfrac{1}{2}k\rho(r)u^2, \qquad k \equiv \frac{C_D A_{\text{ref}}}{m}$$

where air density ρ depends on radial distance only. Aerodynamic area-to-mass ratio k depends on drag coefficient C_D, vehicle mass m, and reference area A_{ref}.

The disturbing acceleration $\boldsymbol{a}$ includes aspherical Earth gravity $\boldsymbol{g}^*$ and kinematic accelerations caused by Earth rotation:

$$\boldsymbol{a} = \boldsymbol{g}^*(\boldsymbol{r}, t) - 2\boldsymbol{\omega}_\oplus \times \boldsymbol{u} - \boldsymbol{\omega}_\oplus \times (\boldsymbol{\omega}_\oplus \times \boldsymbol{r})$$

Disturbing accelerations will be neglected in comparison to aerodynamic and inverse-square gravitational acceleration. By neglecting $\boldsymbol{a}$, the equations of motion are mathematically identical to the nonrotating Earth case. The misalignment of $\boldsymbol{u}$ and $\boldsymbol{v}$ enters through the initial conditions only. First-order perturbations caused by $\boldsymbol{a}$ are needed to accurately predict exoatmospheric orbital motion over many revolutions.

The dynamical and kinematic equations (1) are resolved in different frames, as follows. The dynamical equations (1a) are resolved along aerodynamic axes $\mathbf{U}$, $\mathbf{N}$, and $\mathbf{P}$, respectively, to uncouple lift and drag effects:

$$\dot{u} = -D - \frac{\mu_\oplus}{r^2}\sin\gamma \qquad (2a)$$

$$u\dot{\gamma} = L_v + \left[\frac{u^2}{r} - \frac{\mu_\oplus}{r^2}\right]\cos\gamma \qquad (2b)$$

$$u\cos\gamma\dot{\psi} = L_h - \frac{(u\cos\gamma)^2}{r}\cos\psi\tan\theta \qquad (2c)$$

Flight-path angle γ is measured between the local horizontal plane and $\mathbf{U}$. Heading angle ψ is measured between the projection of $\mathbf{U}$ in the horizontal plane and the $\mathbf{s}$ axis. The kinematic equations (1b) are resolved along $\mathscr{T}$ axes $\mathbf{s}$, $\mathbf{n}$, and

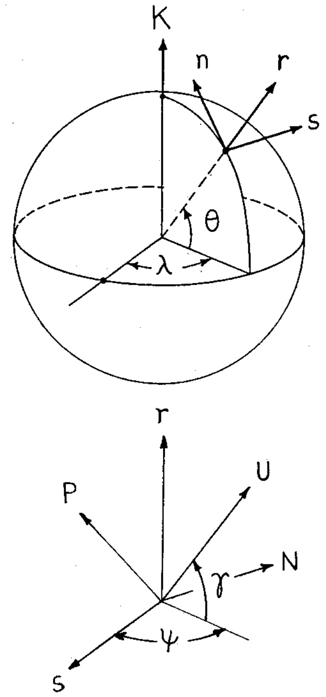

Fig. 1 Topocentric and velocity frames.

$\boldsymbol{r}$ to determine the geocentric position coordinates r, θ, and λ relative to a rotating Earth:

$$u_s = r\dot{\lambda}\cos\theta = u\cos\gamma\cos\psi \tag{3a}$$

$$u_n = r\dot{\theta} = u\cos\gamma\sin\psi \tag{3b}$$

$$u_r = \dot{r} = u\sin\gamma \tag{3c}$$

The sixth-order system defined by Eqs. (2) and (3) is generally nonintegrable.

During exoatmospheric flight, well-known analytic solutions describe $\boldsymbol{r}$ and $\boldsymbol{v}$ (rather than $\boldsymbol{u}$) as functions of an inertial central angle measured in an invariable plane perpendicular to inertial angular momentum $\boldsymbol{r} \times \boldsymbol{v}$. A transcendental equation derived from Kepler's equation is solved for the central angle (true anomaly) as a function of time. Asymptotic matching techniques have been used to combine exoatmospheric Keplerian solutions with approximate endoatmospheric solutions.[8]

During endoatmospheric flight, velocity decreases because of drag while lift torques change the orientation of the orbital plane. The relative angular momentum vector $\boldsymbol{c} = \boldsymbol{r} \times \boldsymbol{u}$ is misaligned from $\boldsymbol{r} \times \boldsymbol{v}$ because of atmospheric rotation. Drag reduces $|\boldsymbol{c}|$, leaving its direction unchanged:

$$\frac{\mathrm{d}\boldsymbol{c}}{\mathrm{d}t} + \frac{D}{u}\boldsymbol{c} = \boldsymbol{r} \times \boldsymbol{L}$$

The plane perpendicular to $\boldsymbol{c}$ is invariant (to zeroth-order) when $\boldsymbol{L} = 0$, since only lift torques can change the direction of $\boldsymbol{c}$. (In reality, the neglected disturbance torques $\boldsymbol{r} \times \boldsymbol{a}$ will change the orientation of this otherwise Earth-fixed plane.)

It will be shown that Eqs. (2) and (3) can be reduced to a second-order system describing motion along a space curve. The space curve could be specified implicitly by four kinematic constraints of the form

$$f_i(r,\theta,\gamma,\psi;\lambda) = 0, \qquad 1 \leq i \leq 4$$

Instead, it is preferable to specify the dependent variables r, θ, γ, and ψ by functions of longitude angle λ rather than time, since the time variable is an ignorable coordinate when $\boldsymbol{a} = 0$. Motion is constrained to the space curve by the lift vector $\boldsymbol{L}$, which is derived from Eq. (2) because trajectory curvature is specified. Velocity magnitude is determined by a quadrature that cannot (in general) be expressed by elementary functions, since drag deceleration is a nonlinear function of velocity and lift magnitude.

Explicit Space Curve Guidance

A three-dimensional space curve is specified by single-valued functions $\zeta(\lambda)$ and $\theta(\lambda)$ of geocentric longitude λ, where ζ is the logarithmic function of radial distance r:

$$\zeta(\lambda) = \ell n\,[r(\lambda)/R_\oplus] \tag{4}$$

and $R_\oplus$ is the Earth's equatorial radius. In the sequel, it will be shown that an *optimal* space curve may be determined by solving a set of coupled, fourth-order, Euler–Lagrange differential equations for ζ and θ.

Flight-path angles do not depend on velocity magnitude in the space curve formulation, which is a significant advantage in the reduction of Eqs. (2) and (3) from sixth- to second-order. When λ is the independent variable, it follows from Eq. (3) that flight-path γ and heading ψ angles are obtained by differentiations of the space curve functions:

$$\tan\psi(\lambda) = \frac{u_n}{u_s} = \frac{1}{\cos\theta}\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} \tag{5a}$$

$$\tan\gamma(\lambda) = \frac{u_r}{(u_s^2 + u_n^2)^{1/2}} = \frac{\cos\psi}{\cos\theta}\frac{d\zeta}{\mathrm{d}\lambda} \tag{5b}$$

The logarithmic function Eq. (4) simplifies the computation of γ.

For looping orbits, $r(\lambda)$ and $\theta(\lambda)$ are not single-valued functions, and a new independent variable, such as arc length (rather than λ), would be required. When arc length is the independent variable, u appears explicitly in the computation of γ and ψ, and reduction to a single degree of freedom is more difficult (if not impossible).

Lift-acceleration commands may be expressed by functions of u^2, $\zeta(\lambda)$, and $\theta(\lambda)$ because the space curve is specified. Compute flight-path angular rates $\dot{\gamma}$ and $\dot{\psi}$ by changing the independent variable from t to λ:

$$\dot{\gamma} = \dot{\lambda}\frac{\mathrm{d}\gamma}{\mathrm{d}\lambda} = \frac{u\cos\gamma\cos\psi}{r\cos\theta}\gamma', \qquad \dot{\psi} = \dot{\lambda}\frac{\mathrm{d}\psi}{\mathrm{d}\lambda} = \frac{u\cos\gamma\cos\psi}{r\cos\theta}\psi'$$

where $\dot{\lambda}$ is given by Eq. (3a) and primes denote differentiation with respect to λ. Substitute these results in Eq. (2) and solve for the lift commands:

$$L_v(u^2,\zeta,\theta,\zeta',\theta',\zeta'',\theta'') = \frac{u^2}{r}\cos\gamma\left(\frac{\cos\psi}{\cos\theta}\gamma' - 1 + \frac{\mu_\oplus}{ru^2}\right) \tag{6a}$$

$$L_h(u^2,\zeta,\theta,\zeta',\theta',\zeta'',\theta'') = \frac{u^2}{r}\left(\frac{\cos^2\gamma\cos\psi}{\cos\theta}\right)(\psi' + \sin\theta) \tag{6b}$$

where γ and ψ introduce the first derivative terms. Desired trajectory shape enters through the curvature terms γ' and ψ', obtained by implicit differentiation of Eq. (5) with respect to λ:

$$\begin{bmatrix}\gamma' \\ \psi'\end{bmatrix} = \begin{bmatrix}\dfrac{\cos^2\gamma\cos\psi}{\cos\theta} & -\zeta'\left(\dfrac{\cos\gamma}{\cos\theta}\right)^2\cos^3\psi\sin\psi \\ 0 & \dfrac{\cos^2\psi}{\cos\theta}\end{bmatrix} \times \begin{bmatrix}\zeta'' + \zeta'\tan\psi\sin\theta \\ \theta'' + \theta'\tan\psi\sin\theta\end{bmatrix} \tag{7}$$

These functions introduce second derivative terms ζ'' and θ''.

Actual lift $\boldsymbol{L}$ lags the lift-acceleration command $\boldsymbol{L}_c$, whose components are specified by Eqs. (6) and (7). For three degree-of-freedom simulations, noninstantaneous response could be modeled by a first-order lag:

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} + \frac{1}{\tau}\boldsymbol{L} = \frac{1}{\tau}\boldsymbol{L}_c$$

where the time constant τ approximates the dominant closed-loop eigenvalue of the autopilot and actuator. In the sequel, instantaneous response ($\tau \to 0$) will be assumed, and it follows that the lift commands of Eqs. (6) are the actual lift components ($\boldsymbol{L} = \boldsymbol{L}_c$).

Lift components contain kinematic terms that are unrelated to the desired space curve shape. L_v compensates for (resultant) gravitational and centrifugal acceleration components perpendicular to velocity. In L_h, the $\sin\theta$ term accounts for an apparent heading change caused by spherical Earth curvature as the vehicle translates. For motion in a great circle plane, L_h is zero because $\psi' = -\sin\theta$. Differentiate the spherical trigonometric identity $\sin\theta = \tan\lambda\tan\psi$ with respect to λ to obtain

$$\cos\theta\,\theta' = \frac{\tan\psi}{\cos^2\lambda} + \frac{\tan\lambda}{\cos^2\psi}\psi'$$

Substitution for θ' using Eq. (5a) and algebraic manipulation of the result establishes the stated conclusion. When $\boldsymbol{a} \neq 0$, lift acceleration is nonzero to offset disturbance (primary coriolis) accelerations perturbing the orbit plane.

Available lift must be sufficient to follow the prescribed space curve. At high altitudes, lift is determined by air density

ρ and lift coefficient:

$$C_L = \frac{(L_v^2 + L_h^2)^{1/2}}{\frac{1}{2}\rho u^2 A_{\text{ref}}/m} = \frac{2m\Lambda}{\rho A_{\text{ref}}} \tag{8a}$$

$$\Lambda = \frac{\cos\gamma}{r}\left[\left(\frac{\cos\psi}{\cos\theta}\gamma' - 1 + \frac{\mu_\oplus}{ru^2}\right)^2 + \left(\frac{\cos\gamma\,\cos\psi}{\cos\theta}\right)^2(\psi' + \sin\theta)^2\right]^{1/2} \tag{8b}$$

C_L is relatively insensitive to velocity magnitude when gravitational and centrifugal accelerations are in equilibrium (Loh's approximation[8]), since $\mu_\oplus/ru^2 - 1$ is (approximately) constant. Admissible space curves satisfy the inequality

$$\Lambda \le \frac{1}{2}\rho C_L^* \frac{A_{\text{ref}}}{m}$$

where C_L^* is the lift coefficient at the maximum (trimmable) angle of attack, both of which are Mach-number dependent in the supersonic and transonic regimes. At low altitudes, structural design considerations limit the maximum normal loading (unmanned vehicles).

Terminal boundary conditions will determine when the lift commands of Eq. (6) exceed the available lift. A four-parameter family of midcourse trajectories is determined by ζ_F, γ_F, θ_F, and ψ_F, since these parameters specify the required trajectory curvature. For fixed-parameter values, $(L_v^2 + L_h^2)^{1/2}$ could be limited to satisfy the constraints, while the direction of $\boldsymbol{L}$ is unchanged. Alternatively, two (or more) of the parameters could be modulated to satisfy command limits on L_v and L_h individually, at each point of the trajectory. In either case, the instantaneous space curves varies until sufficient lift authority is available to follow a particular member of the space curve family to the terminal conditions.

Example Space Curves

The space curve functions $\zeta(\lambda)$ and $\theta(\lambda)$ depend implicitly on at least four constant parameters determined by boundary conditions on position and path angle at maneuver initiation and termination. Thus far, explicit dependence on this parameter set has been suppressed for notational brevity. Example space curves illustrate the importance and usefulness of these parameters in specific applications.

Keplerian Trajectory

A planar trajectory could be defined with the same mathematical function as a Keplerian trajectory:

$$r(\lambda) = p(1 + e\cos\lambda)^{-1}$$

where motion is confined to the equatorial plane ($\theta = \psi = 0$) for simplicity. This geometrical trajectory differs from a true Keplerian ellipse because the central angle λ is measured in a rotating (rather than inertial) frame, with perigee located at $\lambda = 0$. The parameters p and e are analogous to semilatus rectum and eccentricity in the drag-free case.

Although the geometrical curve is an ellipse in the rotating frame, velocity decreases because of drag during the atmospheric encounter. Aerodynamic lift is required to maintain the desired elliptical shape. Lift commands are synthesized by differentiation and substitution of $\theta = \psi = 0$, yielding

$$\zeta' = \tan\gamma = \frac{e\sin\lambda}{1 + e\cos\lambda}, \qquad \gamma' = \frac{e(e + \cos\lambda)}{1 + 2e\cos\lambda + e^2}$$

These results are substituted in Eq. (6) to obtain

$$L_v = \left[\frac{\mu_\oplus}{r^2} - \frac{u^2}{r}\left(\frac{1 + e\cos\lambda}{1 + 2e\cos\lambda + e^2}\right)\right]\cos\gamma, \qquad L_h = 0$$

Based on an earlier discussion, it is clear that $L_h = 0$ because motion occurs in a great circle plane.

As u^2 decreases, L_v increases to maintain the specified γ profile, since gravity would otherwise bend the (nonlifting) trajectory toward the local vertical ($\gamma \to -\pi/2$). It can be shown that $L_v = 0$ in the drag-free case, when

$$u^2 = \mu_\oplus\left[\frac{2}{r} - \frac{1 - e^2}{p}\right], \qquad (L_v = 0)$$

which is similar in form to the well-known energy integral, except that u is the Earth-relative (rather than inertial) velocity. Although u and v differ by 5–7% at perigee velocities ($0 < e < 0.7$), the corresponding energy difference is considerably larger (12–52%, depending on e).

Only two (rather than four) parameters define the planar space curve. These parameters could be specified at atmospheric entry ($\lambda = \lambda_0$):

$$e = \tan\gamma_0[\sin\lambda_0 - \tan\gamma_0\cos\lambda_0]^{-1}, \qquad p = r_0(1 + e\cos\lambda_0)$$

When sufficient lift is available to follow this space curve, atmospheric exit conditions (at $\lambda = \lambda_F$) are identical to the entry conditions resulting from orbit symmetry about perigee. Exit conditions could be independently specified by generalizing $r(\lambda)$ to include two (or more) parameters.

Polynomial Trajectories

A three-dimensional trajectory could be defined by two cubic polynomial functions[1,2]:

$$\theta(\lambda) = \sum_{j=0}^{3} a_j(\lambda - \lambda_0)^j, \qquad \zeta(\lambda) = \sum_{j=0}^{3} b_j(\lambda - \lambda_0)^j$$

Differentiation with respect to λ yields the horizontal and vertical plane path-angle functions:

$$\tan\psi = \frac{1}{\cos\theta}\sum_{j=1}^{3} ja_j(\lambda - \lambda_0)^{j-1}, \quad \tan\gamma = \frac{\cos\psi}{\cos\theta}\sum_{j=1}^{3} jb_j(\lambda - \lambda_0)^{j-1}$$

The eight polynomial coefficients $a_j, b_j (0 \le j \le 3)$ are specified by two sets of boundary conditions: 1) $r_0, \gamma_0, \theta_0, \psi_0$ at the initiation longitude λ_0; and 2) $r_F, \gamma_F, \theta_F, \psi_F$ at the termination longitude λ_F:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \mathbf{F}^{-1}\begin{bmatrix} \theta_0 \\ \tan\psi_0\cos\theta_0 \\ \theta_F \\ \tan\psi_F\cos\theta_F \end{bmatrix}, \qquad \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \mathbf{F}^{-1}\begin{bmatrix} \ln(r_0/R_\oplus) \\ \tan\gamma_0\cos\theta_0/\cos\psi_0 \\ \ln(r_F/R_\oplus) \\ \tan\gamma_F\cos\theta_F/\cos\psi_F \end{bmatrix}$$

$$\mathbf{F} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & \Delta\lambda & \Delta\lambda^2 & \Delta\lambda^3 \\ 0 & 1 & 2\Delta\lambda & 3\Delta\lambda^2 \end{bmatrix}, \qquad \Delta\lambda \equiv \lambda_F - \lambda_0$$

Coefficient singularities occur when $\Delta\lambda = 0$, γ_0, $\gamma_F = \pm\pi/2$, or ψ_0, $\psi_F = \pm\pi/2$.

The qualitative features of cubic space curves are determined by the numerical values of the polynomial coefficients. Based on the condition for three real roots of a cubic polynomial, it can be shown that vertical and horizontal plane trajectories have S-shapes when the polynomial derivatives change algebraic signs at an inflection point, or

$$a_1 < \frac{a_2^2}{3a_3}, \qquad b_1 < \frac{b_2^2}{3b_3}$$

In the specific case of a horizontal plane trajectory

$$a_0 = \theta_0$$

$$a_1 = \tan\psi_0\cos\theta_0$$

$$a_2 = \frac{1}{\Delta\lambda^2}[3(\theta_F - \theta_0) - \Delta\lambda(2\tan\psi_0\cos\theta_0 + \tan\psi_F\cos\theta_F)]$$

$$a_3 = \frac{1}{\Delta\lambda^3}[\Delta\lambda(\tan\psi_F\cos\theta_F + \tan\psi_0\cos\theta_0) - 2(\theta_F - \theta_0)]$$

For example, when $\psi_0 = 0$ and $\theta_0 = \theta_F = 0$, an S-shaped trajectory is guaranteed because $a_1 = 0$, provided $\psi_F \neq 0$. The horizontal plane trajectory crosses its initial plane with nonzero terminal heading by maneuvering out of the initial plane. Similar conclusions pertain to the vertical plane case.

When available lift is sufficient to follow the prescribed space curve, polynomial coefficients may be evaluated at initiation, holding $\Delta\lambda = \lambda_F - \lambda_0$ fixed. In an actual flight, the initial coinditions $\zeta_0, \gamma_0, \theta_0$, and ψ_0 may differ from their nominal values because of insertion errors. When the final conditions are held fixed, polynomial coefficients can be recomputed using the true (rather than nominal) initial conditions. Off-nominal trajectories *converge* to the nominal path because lift commands are modulated to satisfy the original boundary conditions on destination and path angle.

When the lift commands of Eq. (6) exceed vehicle trim capabilities or normal loading limits, lift-command saturation causes the actual values of ζ, γ, θ, and ψ to deviate from the desired values along the space curve. Holding the terminal conditions fixed, the coefficients should be continuously updated with instantaneous (rather than fixed) λ values, allowing $\Delta\lambda = \lambda_F - \lambda$ to vary. As ρ increases, the curvature requirements are satisfied eventually, and the actual trajectory follows that member of the space curve family satisfying the boundary conditions. If command saturation persists throughout the flight, singularities occur when the vehicle arrives at its destination, since $\Delta\lambda = 0$.

Motion Along the Space Curve

A single degree-of-freedom dynamical system specifies velocity u and arc length s along the space curve:

$$\dot{u} + \frac{1}{2} k\rho u^2 = -\frac{\mu_\oplus}{r^2}\sin\gamma, \qquad \dot{s} = u$$

Gravity increases speed along descending flight paths ($\gamma < 0$), while drag always reduces speed. Arc length increases monotonically, since $u > 0$. When $\zeta(\lambda)$ and $\theta(\lambda)$ are specified, it will be shown that velocity may be expressed by a quadrature.

Aerodynamic area-to-mass ratio k is a nonlinear function[6–8,11] of Mach number M, Reynolds number Re, and lift coefficient C_L:

$$k(M,Re,C_L) = C_D(M,Re,C_L)\frac{A_{\text{ref}}}{m}$$

$$C_D(M,Re,C_L) = C_0(M,Re) + C_1(M,Re)C_L + C_2(M,Re)C_L^2$$

$$M = \frac{u}{c_s}, \qquad Re = \frac{ud_{\text{ref}}}{\nu}, \qquad C_L = \frac{2m\Lambda}{\rho A_{\text{ref}}}$$

where $d_{\text{ref}}, A_{\text{ref}}$ are vehicle reference parameters, m is vehicle mass, and Λ is given by Eq. (8). To accurately predict slowdown, it is necessary to model 1) C_D amplification at low Re values[30] for upper atmosphere maneuvers, and 2) nosetip blunting and mass loss due to ablation[29] for lower atmosphere maneuvers. Nosetip blunting effects on C_D introduce 1) dependence on the vehicle bluntness ratio, and 2) an additional state variable measuring the instantaneous nosetip recession.

Atmospheric properties are nonlinear functions of the space curve functions. For example, air density ρ, sound speed c_s, and kinematic viscosity ν depend primarily on altitude, but also on geographic latitude and longitude. Although not included, horizontal wind components are similarly specified.

Instantaneous lift-acceleration magnitude can be modulated to compensate for speed variations δu because of atmospheric and aerodynamic perturbations, while lift direction is unchanged to maintain the same trajectory curvature. For example, when $\delta u < 0$, C_L is slightly larger than nominal to offset the dynamic pressure reduction, but $|\delta u|$ decreases because drag is more sensitive to the latter effect. Alternatively, the terminal boundary conditions could be changed to maintain the same lift acceleration. It is noteworthy that the space curve approach does not require time-to-go because λ is the independent variable.

Solution for $u(\lambda)$ may be expressed by a quadrature because the nonlinear function $k(M,Re,C_L)$ may be expressed, ultimately, as a function of u and λ. From the preceding discussion, it follows that aerodynamic and atmospheric properties have the (intermediate) functional forms:

$$k(M,Re,C_L) = k(u^2,\zeta,\theta,\zeta',\theta',\zeta'',\theta''), \qquad \rho = \rho(\zeta)$$

The independent variable λ enters implicitly through $\zeta(\lambda)$ and $\theta(\lambda)$ in the lift-induced drag term and the altitude-dependent functions ρ, c_s, and ν. Change the independent variable from t to λ:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(u^2) + \beta(\lambda,u^2)u^2 = g(\lambda) \tag{9a}$$

$$\beta(\lambda,u^2) \equiv \frac{r\rho\cos\theta}{\cos\gamma\cos\psi}k(\lambda,u^2) \tag{9b}$$

$$k(\lambda,u^2) = \frac{A_{\text{ref}}}{m}[C_0(\lambda,u^2) + C_1(\lambda,u^2)C_L + C_2(\lambda,u^2)C_L^2] \tag{9c}$$

$$g(\lambda) \equiv -\frac{2\mu_\oplus\cos\theta\tan\gamma}{r\cos\psi} \tag{9d}$$

where $g > 0$ for descending trajectories ($\gamma < 0$). Although β and g are really *implicit* functions of λ, it is understood that β and g become explicit functions of λ when $\zeta(\lambda),\theta(\lambda)$ and their derivatives are specified.

The first-order, nonlinear differential equation (9a) may be integrated by the method of successive approximation. This technique was successfully applied to predict velocity and flight-path angle histories for nonlifting trajectories through the transonic regime and transition to vertical descent.[25–27] Closed-form solution is precluded by the nonlinear coupling between u^2 and λ in the drag term β.

The zeroth-order solution will be obtained in the hypersonic limit ($u^2 \to \infty$), when k depends primarily on C_L:

$$k_\infty(\lambda) = \frac{A_{\text{ref}}}{m}[\bar{C}_0 + \bar{C}_1 C_L(\lambda,\infty) + \bar{C}_2 C_L^2(\lambda,\infty)]$$

$$C_L(\lambda,\infty) = \frac{2m}{\rho A_{\text{ref}}}\frac{\cos\gamma}{r}\left[\left(\frac{\cos\psi}{\cos\theta}\gamma' - 1\right)^2 + \left(\frac{\cos\gamma\cos\psi}{\cos\theta}\right)^2(\psi' + \sin\theta)^2\right]^{1/2}$$

At high velocities, C_L depends primarily on λ, since dependence on u^2 is important at low velocities because of gravity bending effects. At infinite M and Re values, the functions $\bar{C}_0$, $\bar{C}_1$, and $\bar{C}_2$ are approximately constants. For example, an empirical inviscid model[26] for C_0 predicts $C_0 \to C_\infty$ (a constant) as $M \to \infty$:

$$C_0(M) = C_\infty[1 + \mu\exp(\Sigma c_m\mu^m)], \qquad \mu = \frac{1}{M^2}$$

where the constants C_∞ and c_m ($m \geq 0$) can be fitted to actual drag-coefficient data at $C_L = 0$.

Reduction to quadrature is straightforward in the hypersonic limit because $\beta(\lambda,u^2)$ depends only on the independent variable:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(u_0^2) + \beta_0(\lambda)u_0^2 = g(\lambda)$$

$$\beta_0(\lambda) = \frac{r\rho\cos\theta}{\cos\gamma\cos\psi}k_\infty(\lambda)$$

where $g(\lambda)$ is specified by Eq. (9d). Assume a solution of the form

$$u_0^2(\lambda) = A_0(\lambda)e^{-\phi_0(\lambda)}$$

Substitution in the differential equation results in a set of uncoupled, first-order equations for A_0 and ϕ_0:

$$\phi_0' = \beta_0(\lambda), \qquad A_0' = g(\lambda)e^{\phi_0}$$

Integrate to obtain the zeroth-order approximation:

$$u_0^2(\lambda) = \left[u^2(0) + \int_{\lambda_0}^{\lambda} g(\lambda)e^{\phi_0}\,\mathrm{d}\lambda\right]e^{-\phi_0} \tag{10a}$$

$$\phi_0(\lambda) = \int_{\lambda_0}^{\lambda} \beta_0(\lambda)\,\mathrm{d}\lambda \tag{10b}$$

where $u(0)$ is the initial velocity magnitude.

Successive approximations $(i \geq 1)$ are generated by the recursion formula.

$$u_i^2(\lambda) = \left[u^2(0) + \int_{\lambda_0}^{\lambda} g(\lambda)e^{\phi_i}\,\mathrm{d}\lambda\right]e^{-\phi_i} \tag{11a}$$

$$\phi_i(\lambda) = \int_{\lambda_0}^{\lambda} \beta_{i-1}(\lambda)\,\mathrm{d}\lambda \tag{11b}$$

where $k(\lambda,u^2)$ is evaluated using the $(i-1)$st velocity approximation:

$$\beta_{i-1}(\lambda) = \frac{r\rho\cos\theta}{\cos\gamma\cos\psi}\,k[u_{i-1}^2(\lambda),\lambda] \tag{11c}$$

Space curve geometry is invariant during each iteration because the functions describing the shape are unchanged, provided lift-command saturation does not occur. Successive approximations of Eq. (11) improve the accuracy of the slowdown predicted by the nonlinear drag law. Convergence is rapid (two iterations) when drag coefficient is relatively constant (in the hypersonic and high supersonic regimes). Slower convergence occurs at transonic Mach numbers when compressibility effects magnify C_D. In the latter case, it may not be possible to generate sufficient lift at the maximum trimmable angle of attack, necessitating a different approach.

Time-of-flight along the space curve is expressed by a quadrature following solution for $u(\lambda)$:

$$t = t_0 + \int_{\lambda_0}^{\lambda} \frac{r\cos\theta}{u(\lambda)\cos\gamma\cos\psi}\,\mathrm{d}\lambda$$

where t_0 is the initial time. Although the integrand depends only on λ, the quadrature must be evaluated numerically because $u(\lambda)$ is specified by a numerical (rather than analytic) solution. Singularities occur during vertical descent $(\gamma = -\pi/2)$ or when longitude is stationary $(\psi = \pm\pi/2)$. Arc length is expressed by a quadrature that does not involve $u(\lambda)$:

$$s(\lambda) = s(\lambda_0) + \int_{\lambda_0}^{\lambda} \frac{r\cos\theta}{\cos\gamma\cos\psi}\,\mathrm{d}\lambda$$

This solution is useful when arc length, rather than time, defines position along the trajectory.

In the drag-free case $(\beta = 0)$, velocity is a known function of position along the trajectory because an energy integral exists:

$$u^2(\lambda) - \int_{\lambda_0}^{\lambda} g(\lambda)\,\mathrm{d}\lambda = 2E$$

where E is constant along the trajectory. Using the kinematic identity of Eq. (5b), it can be shown that the integrand is an exact differential:

$$g(\lambda)\,\mathrm{d}\lambda = 2\,\mathrm{d}(\mu_\oplus/r)$$

and the quadrature may be evaluated in closed-form:

$$\tfrac{1}{2}u^2 - (\mu_\oplus/r) = E$$

An energy integral exists when disturbance accelerations are included $(\boldsymbol{a} \neq 0)$, except that 1) relative velocity $\boldsymbol{u}$ must be replaced with the inertial velocity v, and 2) the gravitational acceleration $\boldsymbol{g}^*$ must be independent of time.

Space Curve Optimization

An optimal, three-dimensional space curve is the solution of a two-point boundary-value problem in the calculus of variations.[31] With the space curve approach, the optimization problem can be formulated with a single state variable $x = u^2$, since time (or arc length) is generally an ignorable coordinate. Space curve functions $\zeta(\lambda)$ and $\theta(\lambda)$ are regarded as *control variables* that will be chosen to optimize the cost function:

$$J = \Phi(x_F,\lambda_F) + \int_{\lambda_0}^{\lambda_F} \mathscr{L}(\zeta,\zeta',\zeta'',\theta,\theta',\theta'',x,\lambda)\,\mathrm{d}\lambda$$

Primes denote differentiation with respect to λ, which is assumed to change monotonically (i.e., no loops). Φ specifies the terminal state x_F at the prescribed destination λ_F. Finally, $\mathscr{L}$ is a continuously differentiable, single-valued function of its eight arguments.

The space curve optimization problem is unlike conventional optimization problems[11,32] because the cost function and state equation depend on the first and second derivatives of the control variables through lift acceleration. For example, maneuver impulse (or integral-square lift) can be minimized using a cost function:

$$\mathscr{L} = L_v^2 + L_h^2 = x^2\Lambda^2(\zeta,\zeta',\zeta'',\theta,\theta',\theta'')$$

where Λ is defined by Eq. (8). For lifting trajectories, the state equation always depends on control variable derivatives:

$$x' = f(\zeta,\zeta',\zeta'',\theta,\theta',\theta'',x) = -\beta(\zeta,\zeta',\zeta'',\theta,\theta',\theta'',x)x + g(\zeta,\zeta',\theta,\theta')$$

where the initial value $x(\lambda_0)$ is specified. In the optimization problem, β and g are *implicit* functions of λ, which enters through the (unknown) functions $\zeta(\lambda),\theta(\lambda)$, and their derivatives. (In the preceding discussion, β and g were treated as explicit functions of λ to reduce Eq. (9) to a quadrature.)

Necessary conditions for optimality are 1) a set of coupled, fourth-order, Euler–Lagrange differential equations for the space curve functions; 2) a single, first-order, Euler–Lagrange equation for an adjoint variable; and 3) boundary conditions specified at fixed endpoints λ_0 and λ_F. Necessary conditions are derived from the first variation of the augmented cost function (refer to the Appendix):

$$J = \Phi + \int_{\lambda_0}^{\lambda_F} (\mathscr{H} - \chi x')\,\mathrm{d}\lambda, \qquad \mathscr{H} \equiv \mathscr{L} + \chi f$$

where χ is a Lagrange-multiplier function. In the nonautonomous case, the Hamiltonian function $\mathscr{H}$ depends explicitly on the independent variable $(\mathscr{H}_\lambda \neq 0)$, thereby precluding a first integral. The second variation must be computed to determine if the extremum is a maximum or a minimum.

Extremal space curves are solutions of the Euler–Lagrange equations:

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\left(\frac{\partial\mathscr{H}}{\partial\zeta''}\right) - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathscr{H}}{\partial\zeta'}\right) + \frac{\partial\mathscr{H}}{\partial\zeta} = 0 \tag{12}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\left(\frac{\partial\mathscr{H}}{\partial\theta''}\right) - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathscr{H}}{\partial\theta'}\right) + \frac{\partial\mathscr{H}}{\partial\theta} = 0 \tag{13}$$

When $\mathscr{H}$ does not depend on derivatives of the control variables, the necessary condition for optimality is an algebraic (rather than differential) equation. In the Appendix, it is shown that Eqs. (12) and (13) are *fourth-order*, nonlinear differential equations specifying ζ'''' and θ'''', with sets of boundary conditions specified at trajectory initiation λ_0 and termination λ_F:

$$\mathscr{C}_0(\lambda_0) = [\zeta(\lambda_0),\zeta'(\lambda_0),\theta(\lambda_0),\theta'(\lambda_0)]$$

$$\mathscr{C}_F(\lambda_F) = [\zeta(\lambda_F),\zeta'(\lambda_F),\theta(\lambda_F),\theta'(\lambda_F)]$$

The Lagrange multiplier is the solution of an adjoint differential equation that is linear and homogeneous in χ:

$$\chi' = -\frac{\partial \mathscr{H}}{\partial x} = -\left(\frac{\partial \mathscr{L}}{\partial x} + \chi\frac{\partial f}{\partial x}\right), \qquad \chi(\lambda_F) = \frac{\partial \Phi}{\partial x_F} \tag{14}$$

When x and the space curve functions are specified, Eq. (14) admits a unique solution for χ with boundary condition $\chi(\lambda_F)$ specified at trajectory termination (rather than initiation). It is unlikely that $\chi' = 0$ because f always depends on velocity $[(\partial f/\partial x) \neq 0]$.

Optimal space curve functions are assumed to have prescribed values $\mathscr{C}_F(\lambda_F)$ at trajectory termination λ_F. For certain problems (e.g., maximum range), these boundary values may not be specified, or the termination point may be determined by more general constraints, such as a boundary curve or surface in parameter space. In addition to finding the optimal space curve, the region of integration of the independent variable must be determined. When λ_F is specified but $\mathscr{C}_F$ is unspecified, the following *natural* boundary conditions must be satisfied:

$$\frac{\partial \mathscr{H}}{\partial \zeta''} = 0, \qquad \frac{\partial \mathscr{H}}{\partial \zeta'} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathscr{H}}{\partial \zeta''}\right) = 0 \qquad \text{at } \lambda = \lambda_F$$

$$\frac{\partial \mathscr{H}}{\partial \theta''} = 0, \qquad \frac{\partial \mathscr{H}}{\partial \theta'} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathscr{H}}{\partial \theta''}\right) = 0 \qquad \text{at } \lambda = \lambda_F$$

When λ_F is not specified, a *transversality condition* must be satisfied:

$$\frac{\partial \Phi}{\partial \lambda_F} + \mathscr{H}_F = 0 \qquad \text{at } \lambda = \lambda_F$$

The original equations of motion [Eqs. (2) and (3)] depend on λ implicitly, through $\zeta(\lambda),\theta(\lambda)$ and their derivatives. Explicit dependence on λ can occur when 1) disturbing accelerations are included, or 2) atmospheric properties depend on λ. Important simplifications occur when $\mathscr{L}$ and f are longitudinally symmetric (i.e., autonomous):

$$\frac{\partial \mathscr{L}}{\partial \lambda} = 0, \qquad \frac{\partial f}{\partial \lambda} = 0$$

The autonomous problem admits a first integral of motion, not necessarily the same as the Hamiltonian function:

$$\mathscr{F} = \zeta'\left[\frac{\partial \mathscr{H}}{\partial \zeta'} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathscr{H}}{\partial \zeta''}\right)\right] + \zeta''\frac{\partial \mathscr{H}}{\partial \zeta''} + \theta'\left[\frac{\partial \mathscr{H}}{\partial \theta'} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathscr{H}}{\partial \theta''}\right)\right] + \theta''\frac{\partial \mathscr{H}}{\partial \theta''} - \mathscr{H} = c \tag{15}$$

It follows that $\mathscr{F} = -\mathscr{H}$ only if $\mathscr{H}$ depends on two (rather than eight) variables, x and χ (all other partials zero). The constant c is evaluated using the boundary conditions, which is not straightforward because they are specified at two different points. When $\mathscr{C}_F$ and λ_F are unspecified, the natural boundary and transversality conditions specify $c = \partial \Phi/\partial \lambda_F$.

The first integral is another linear, first-order, differential equation for $\chi(\lambda)$ because $\mathscr{H}$ and its partial derivatives are linear in χ. When c is specified, χ' can be eliminated from Eq. (15) and adjoint equation (14), yielding an explicit solution for $\chi(\lambda)$:

$$\chi(\lambda) = -\frac{A\dfrac{\partial \mathscr{L}}{\partial x} + C}{B + A\dfrac{\partial f}{\partial x}}$$

$$A \equiv \zeta'\frac{\partial f}{\partial \zeta''} + \theta'\frac{\partial f}{\partial \theta''}$$

$$B \equiv \zeta'\left[\frac{\partial f}{\partial \zeta'} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial f}{\partial \zeta''}\right)\right] + \zeta''\frac{\partial f}{\partial \zeta''} + \theta'\left[\frac{\partial f}{\partial \theta'} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial f}{\partial \theta''}\right)\right] + \theta''\frac{\partial f}{\partial \theta''} - f$$

$$C \equiv \zeta'\left[\frac{\partial \mathscr{L}}{\partial \zeta'} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathscr{L}}{\partial \zeta''}\right)\right] + \zeta''\frac{\partial \mathscr{L}}{\partial \zeta''} + \theta'\left[\frac{\partial \mathscr{L}}{\partial \theta'} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathscr{L}}{\partial \theta''}\right)\right] + \theta''\frac{\partial \mathscr{L}}{\partial \theta''} - (\mathscr{L} + c)$$

As the state $x = u^2$ (asymptotically) approaches terminal speed (drag = gravity), χ becomes exponentially large, since the denominator approaches zero with f and its partial derivatives. Under these conditions, performance predictions will be extremely sensitive to boundary conditions.

Numerical Considerations

An optimal space curve is determined numerically by solving a two-point boundary-value problem. The numerical solution will specify solution sets at discrete points $\lambda_0 \leq \lambda_i \leq \lambda_F$:

$$\mathscr{S}_v(\lambda_i) = [\zeta(\lambda_i),\zeta'(\lambda_i),\zeta''(\lambda_i),\zeta'''(\lambda_i)]$$

$$\mathscr{S}_n(\lambda_i) = [\theta(\lambda_i),\theta'(\lambda_i),\theta''(\lambda_i),\theta'''(\lambda_i)]$$

Only the first three elements of $\mathscr{S}_v$ and $\mathscr{S}_n$ are required to generate lift commands at λ_i using Eqs. (6) and (7).

An optimal curve will depend (parametrically) on the specified boundary conditions $\mathscr{C}_0$ and $\mathscr{C}_F$. Although $\mathscr{C}_F$ will not change because end conditions are generally fixed, off-nominal initial conditions are likely because of insertion errors. A four-parameter family (with respect to $\mathscr{C}_0$) of optimal solutions is required. This problem may be simplified by expressing optimal solutions as power series in λ, as in the case of cubic polynomial trajectories (discussed earlier).

In the "shooting method," the two-point boundary-value problem is reduced to a problem of determining the zeros of a system of nonlinear algebraic equations.[33] Partition the ten-element state vector $\mathbf{z}$ based on the known and unknown initial conditions:

$$\mathbf{z}(\lambda) = [\mathbf{v},\mathbf{p}]^T, \quad \mathbf{v}(\lambda) = [\zeta,\zeta',\theta,\theta',x]^T, \quad \mathbf{p}(\lambda) = [\zeta'',\zeta''',\theta'',\theta''',\chi]^T$$

Although $\mathbf{v}(\lambda_0)$ is specified, the unknown parameter vector $\mathbf{p}(\lambda_0)$ must be determined to satisfy specified end conditions:

$$\mathbf{w}(\lambda) = [\zeta,\zeta',\theta,\theta',\chi]^T$$

The Euler–Lagrange equations (12) and (13), adjoint equation (14), and state equation (9) define an initial-value problem whose solutions are *linear* with respect to $\mathbf{p}$:

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}\lambda} = \tilde{\mathbf{F}}(\mathbf{z},\lambda), \qquad \mathbf{z}(\lambda_0) = [\mathbf{v}(\lambda_0),\mathbf{p}(\lambda_0)]^T$$

Generate a five-parameter family of solutions by changing each element of $\mathbf{p}$ individually, holding $\mathbf{v}(\lambda_0)$ fixed, and derive a 25-element matrix $\partial\mathbf{w}/\partial\mathbf{p}$ of influence coefficients. Newton's method is used to iteratively converge to a solution satisfying the desired end conditions $\mathbf{w}(\lambda_F)$:

$$\mathbf{p}(\lambda_0) = \tilde{\mathbf{p}}(\lambda_0) + \left[\frac{\partial \mathbf{w}}{\partial \mathbf{p}}\right]^{-1} [\mathbf{w}(\lambda_F) - \tilde{\mathbf{w}}(\lambda_F)]$$

where $\tilde{\mathbf{w}}$ is the predicted vector corresponding to $\tilde{\mathbf{p}}$.

Conclusions

An explicit midcourse guidance theory was developed for unthrusted, modulated-lift entry vehicles maneuvering to a prescribed destination with terminal constraints on velocity vector direction. Geometrical trajectory shape was specified by two functions expressing altitude and geocentric latitude by functions of Earth-fixed longitude. Two differentiations of each trajectory function determine slopes and curvatures, thus specifying velocity flight-path angles, Eq. (5), and lift-acceleration components, Eq. (6), for acceleration-command autopilot inputs. During periods of lift-command saturation, the instantaneous space curve parameters vary until sufficient lift is available to follow an individual member of a family of space curves.

Although energy is not conserved because of drag, motion along the space curve is integrable because lift-induced drag is determined by trajectory curvature. Velocity along the space curve may be expressed by a quadrature, Eq. (11), evaluated by the method of successive approximation, since nonlinearities preclude closed-form expressions involving elementary functions. Space curve geometry is invariant on each iteration, provided lift-command saturation does not occur.

Optimal space curve functions can be determined by solving a two-point boundary-value problem in the calculus of variations. Necessary conditions for optimality include 1) a set of coupled, fourth-order Euler–Lagrange differential equations for the space curve functions, 2) a single, first-order Euler–Lagrange equation for the adjoint variable corresponding to the single state variable (velocity), and 3) boundary conditions specified at initiation and termination. When the Hamiltonian is not explicitly dependent on longitude (the independent variable), the existence of a first integral, not necessarily the Hamiltonian function, allows analytic solution for the adjoint variable.

Appendix

The control variables ζ,θ will be chosen to optimize the cost function:

$$J = \Phi(x_F,\lambda_F) + \int_{\lambda_0}^{\lambda_F} \mathscr{L}(\zeta,\zeta',\zeta'',\theta,\theta',\theta'',x,\lambda)\,\mathrm{d}\lambda$$

The single state variable $x = u^2$ is determined by

$$x' = f(\zeta,\zeta',\zeta'',\theta,\theta',\theta'',x,\lambda) = -\beta x + g, \qquad x(\lambda_0) = x_0$$

where the initial value $x(\lambda_0)$ is specified. Adjoin the state equation to the cost function with a Lagrange-multiplier function $\chi(\lambda)$:

$$J = \Phi + \int_{\lambda_0}^{\lambda_F} (\mathscr{H} - \chi x')\,\mathrm{d}\lambda, \qquad \mathscr{H} \equiv \mathscr{L} + \chi f$$

Integrate by parts to obtain

$$J = \Phi - \chi(\lambda_F)x(\lambda_F) + \chi(\lambda_0)x(\lambda_0) + \int_{\lambda_0}^{\lambda_F} (\mathscr{H} + \chi' x)\,\mathrm{d}\lambda$$

Necessary Conditions

Necessary conditions for optimality are derived from the first variation of the augmented cost function, expressed by a sum of three terms:

$$\delta J = \delta J_s + \delta J_v + \delta J_h$$

representing (independent) variations with respect to 1) the state variable

$$\delta J_s = \left(\frac{\partial \Phi}{\partial x} - \chi\right)\delta x(\lambda_F) + \chi(\lambda_0)\delta x(\lambda_0) + \int_{\lambda_0}^{\lambda_F} \left(\frac{\partial \mathscr{H}}{\partial x} + \chi'\right)\delta x\,\mathrm{d}\lambda$$

2) the vertical space curve function $\zeta(\lambda)$ and its derivatives

$$\delta J_v = \int_{\lambda_0}^{\lambda_F} \left[\frac{\partial \mathscr{H}}{\partial \zeta}\delta\zeta + \frac{\partial \mathscr{H}}{\partial \zeta'}\delta\zeta' + \frac{\partial \mathscr{H}}{\partial \zeta''}\delta\zeta''\right]\mathrm{d}\lambda$$

$$= \delta\mathscr{B}_v(\lambda_F) - \delta\mathscr{B}_v(\lambda_0) + \delta I_v$$

$$\delta\mathscr{B}_v(\lambda) = \frac{\partial \mathscr{H}}{\partial \zeta''}\delta\zeta' + \left[\frac{\partial \mathscr{H}}{\partial \zeta'} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathscr{H}}{\partial \zeta''}\right)\right]\delta\zeta$$

$$\delta I_v = \int_{\lambda_0}^{\lambda_F} \left[\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\left(\frac{\partial \mathscr{H}}{\partial \zeta''}\right) - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathscr{H}}{\partial \zeta'}\right) + \frac{\partial \mathscr{H}}{\partial \zeta}\right]\delta\zeta\,\mathrm{d}\lambda$$

3) the horizontal space curve function $\theta(\lambda)$ and its derivatives

$$\delta J_h = \int_{\lambda_0}^{\lambda_F} \left[\frac{\partial \mathscr{H}}{\partial \theta}\delta\theta + \frac{\partial \mathscr{H}}{\partial \theta'}\delta\theta' + \frac{\partial \mathscr{H}}{\partial \theta''}\delta\theta''\right]\mathrm{d}\lambda$$

$$= \delta\mathscr{B}_h(\lambda_F) - \delta\mathscr{B}_h(\lambda_0) + \delta I_h$$

$$\delta\mathscr{B}_h(\lambda) = \frac{\partial \mathscr{H}}{\partial \theta''}\partial\theta' + \left[\frac{\partial \mathscr{H}}{\partial \theta'} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathscr{H}}{\partial \theta''}\right)\right]\delta\theta$$

$$\delta I_h = \int_{\lambda_0}^{\lambda_F} \left[\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\left(\frac{\partial \mathscr{H}}{\partial \theta''}\right) - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathscr{H}}{\partial \theta'}\right) + \frac{\partial \mathscr{H}}{\partial \theta}\right]\delta\theta\,\mathrm{d}\lambda$$

The necessary condition for an extremal curve is

$$\delta J = \delta J_s + \delta J_v + \delta J_h = 0$$

Each variation must be zero individually, since the three are independent:

$$\delta J_s = 0, \qquad \delta J_v = 0, \qquad \delta J_h = 0$$

Necessary conditions for $\delta J_s = 0$ are an adjoint differential equation for χ and its boundary condition

$$\chi' = -\frac{\partial \mathscr{H}}{\partial x} = -\left(\frac{\partial \mathscr{L}}{\partial x} + \chi\frac{\partial f}{\partial x}\right), \qquad \chi(\lambda_F) = \frac{\partial \Phi}{\partial x_F}$$

where $\delta x(\lambda_0) = 0$ because $x(\lambda_0)$ is specified. Necessary conditions for $\delta J_v = 0$ and $\delta J_h = 0$ are a system of nonlinear, ordinary differential equations for $\zeta(\lambda)$ and $\theta(\lambda)$, and their boundary conditions

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\left(\frac{\partial \mathscr{H}}{\partial \zeta''}\right) - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathscr{H}}{\partial \zeta'}\right) + \frac{\partial \mathscr{H}}{\partial \zeta} = 0$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\left(\frac{\partial \mathscr{H}}{\partial \theta''}\right) - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathscr{H}}{\partial \theta'}\right) + \frac{\partial \mathscr{H}}{\partial \theta} = 0$$

since $\delta I_v = 0$ and $\delta I_h = 0$ when their respective integrands are zero.

Boundary terms $\delta\mathscr{B}_v$ and $\delta\mathscr{B}_h$ do not contribute to the first variation when the space curve functions and their derivatives are specified at fixed endpoints, since the corresponding variations are zero:

$$\delta\zeta(\lambda_0) = \delta\zeta'(\lambda_0) = \delta\theta(\lambda_0) = \delta\theta'(\lambda_0) = 0$$

$$\delta\zeta(\lambda_F) = \delta\zeta'(\lambda_F) = \delta\theta(\lambda_F) = \delta\theta'(\lambda_F) = 0$$

The eight boundary conditions required to integrate the Euler–Lagrange equations are

$$\zeta(\lambda_0),\quad \zeta'(\lambda_0),\quad \theta(\lambda_0),\quad \theta'(\lambda_0)\qquad \text{at } \lambda=\lambda_0$$

$$\zeta(\lambda_F),\quad \zeta'(\lambda_F),\quad \theta(\lambda_F),\quad \theta'(\lambda_F)\qquad \text{at } \lambda=\lambda_F$$

When λ_F is specified, but the terminal values of ζ and θ (and derivatives) are unspecified, the boundary variations $\delta\mathscr{B}_v(\lambda_F)$ and $\delta\mathscr{B}_h(\lambda_F)$ are zero when

$$\frac{\partial \mathscr{H}}{\partial \zeta''}=0,\qquad \frac{\partial \mathscr{H}}{\partial \zeta'}-\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathscr{H}}{\partial \zeta''}\right)=0\qquad \text{at } \lambda=\lambda_F$$

$$\frac{\partial \mathscr{H}}{\partial \theta''}=0,\qquad \frac{\partial \mathscr{H}}{\partial \theta'}-\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathscr{H}}{\partial \theta''}\right)=0\qquad \text{at } \lambda=\lambda_F$$

since $\delta\zeta(\lambda_F)$, $\delta\zeta'(\lambda_F)$, $\delta\theta(\lambda_F)$, and $\delta\theta'(\lambda_F)$ can be arbitrary (nonzero). No modifications of the Euler–Lagrange equations are necessary.

When the final range λ_F is not specified, it is necessary to include variations $\delta\lambda_F$ in this parameter:

$$\delta^* J=\frac{\partial J}{\partial \lambda_F}\delta\lambda_F=\left[\frac{\partial \Phi}{\partial x_F}x'+\frac{\partial \Phi}{\partial \lambda_F}+\mathscr{L}\right]\delta\lambda_F$$

For arbitrary $\delta\lambda_F$, it is clear that $\delta^* J=0$ when the *transversality condition* is satisfied:

$$\frac{\partial \Phi}{\partial \lambda_F}=\mathscr{H}_F=0$$

where the adjoint boundary condition ($\chi_F=\partial\Phi/\partial x_F$) was used.

Explicit Forms

Necessary conditions for extremal space curves are a set of coupled, fourth-order, nonlinear Euler–Lagrange differential equations, derived by performing the indicated differentiations. In order to simplify the notation, partial derivatives of $\mathscr{H}$ will be denoted by numerical subscripts, as follows:

$$\mathscr{H}_1=\frac{\partial \mathscr{H}}{\partial \lambda},\quad \mathscr{H}_2=\frac{\partial \mathscr{H}}{\partial \zeta},\quad \mathscr{H}_3=\frac{\partial \mathscr{H}}{\partial \zeta'},\quad \mathscr{H}_4=\frac{\partial \mathscr{H}}{\partial \zeta''}$$

$$\mathscr{H}_5=\frac{\partial \mathscr{H}}{\partial \theta},\quad \mathscr{H}_6=\frac{\partial \mathscr{H}}{\partial \theta'},\quad \mathscr{H}_7=\frac{\partial \mathscr{H}}{\partial \theta''},\quad \mathscr{H}_8=\frac{\partial \mathscr{H}}{\partial x}$$

With this shorthand notation, the two Euler–Lagrange equations are

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}(\mathscr{H}_i)-\frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathscr{H}_j)+\mathscr{H}_k=0$$

where $i=4, j=3, k=2$ for the vertical plane problem and $i=7, j=6, k=5$ for the horizontal plane problem. From the definition of the Hamiltonian, it follows that

$$\mathscr{H}_1=\mathscr{L}_1+\chi(\lambda)f_1+\chi' f$$

$$\mathscr{H}_i=\mathscr{L}_i+\chi(\lambda)f_i,\qquad 2\le i\le 8$$

where χ is treated as a dependent variable. Compute the total derivative using the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathscr{H}_i)=\frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathscr{L}_i)+\chi\frac{\mathrm{d}}{\mathrm{d}\lambda}(f_i)+\chi' f_i$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathscr{L}_i)=\mathscr{L}_{i1}+\mathscr{L}_{i2}\zeta'+\mathscr{L}_{i3}\zeta''+\mathscr{L}_{i4}\zeta'''+\mathscr{L}_{i5}\theta'$$
$$+\mathscr{L}_{i6}\theta''+\mathscr{L}_{i7}\theta'''+\mathscr{L}_{i8}f$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(f_i)=f_{i1}+f_{i2}\zeta'+f_{i3}\zeta''+f_{i4}\zeta'''+f_{i5}\theta'+f_{i6}\theta''+f_{i7}\theta'''+f_{i8}f$$

Collecting terms, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathscr{H}_i)=\mathscr{H}_{i1}+\mathscr{H}_{i2}\zeta'+\mathscr{H}_{i3}\zeta''+\mathscr{H}_{i4}\zeta'''+\mathscr{H}_{i5}\theta'$$
$$+\mathscr{H}_{i6}\theta''+\mathscr{H}_{i7}\theta'''+\mathscr{H}_{i8}f$$

By a similar procedure, the second derivative may be expressed by

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}(\mathscr{H}_i)=\frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathscr{H}_{i1})+\mathscr{H}_{i8}\frac{\mathrm{d}f}{\mathrm{d}\lambda}+\frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathscr{H}_{i8})f$$
$$+\mathscr{H}_{i4}\zeta''''+\left[\mathscr{H}_{i3}+\frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathscr{H}_{i4})\right]\zeta'''+\left[\mathscr{H}_{i2}+\frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathscr{H}_{i3})\right]\zeta''$$
$$+\frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathscr{H}_{i2})\zeta'+\mathscr{H}_{i7}\theta''''+\left[\mathscr{H}_{i6}+\frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathscr{H}_{i7})\right]\theta'''$$
$$+\left[\mathscr{H}_{i5}+\frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathscr{H}_{i6})\right]\theta''+\frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathscr{H}_{i5})\zeta'$$

The following total derivative terms introduce nonlinearities in the space curve derivatives ($1\le m\le 8$):

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathscr{H}_{im})=\mathscr{H}_{im1}+\mathscr{H}_{im2}\zeta'+\mathscr{H}_{im3}\zeta''+\mathscr{H}_{im4}\zeta'''+\mathscr{H}_{im5}\theta'$$
$$+\mathscr{H}_{im6}\theta''+\mathscr{H}_{im7}\theta'''+\mathscr{H}_{im8}f$$

Collecting terms, the two Euler–Lagrange equations are

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}(\mathscr{H}_i)-\frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathscr{H}_j)+\mathscr{H}_k=\mathscr{H}_{i11}+\mathscr{H}_{i8}f_1-\mathscr{H}_{j1}+\mathscr{H}_k$$
$$+\mathscr{H}_{i4}\zeta''''+(\mathscr{H}_{i3}+2\mathscr{H}_{i41}-\mathscr{H}_{j4}+f_4\mathscr{H}_{i8})\zeta'''+(\mathscr{H}_{i2}$$
$$+2\mathscr{H}_{i31}-\mathscr{H}_{j3}+f_3\mathscr{H}_{i8})\zeta''+\mathscr{H}_{i7}\theta''''+(\mathscr{H}_{i6}+2\mathscr{H}_{i71}$$
$$-\mathscr{H}_{j7}+f_7\mathscr{H}_{i8})\theta'''+(\mathscr{H}_{i5}+2\mathscr{H}_{i61}-\mathscr{H}_{j6}+f_6\mathscr{H}_{i8})\theta''$$
$$+(2\mathscr{H}_{i21}-\mathscr{H}_{j2}+f_2\mathscr{H}_{18})\zeta'+(2\mathscr{H}_{i51}-\mathscr{H}_{j5}+f_5\mathscr{H}_{i8})\theta'$$
$$+(\mathscr{H}_{i8}f_8+2\mathscr{H}_{i81}-\mathscr{H}_{j8})f+N^{(i)}=0$$

$$N^{(i)}\equiv[\zeta'\zeta''\zeta'''\theta'\theta''\theta'''f]$$
$$\begin{bmatrix}\mathscr{H}_{i22}&\mathscr{H}_{i23}&\mathscr{H}_{i24}&\mathscr{H}_{i25}&\mathscr{H}_{i26}&\mathscr{H}_{i27}&\mathscr{H}_{i28}\\ \mathscr{H}_{i32}&\mathscr{H}_{i33}&\mathscr{H}_{i34}&\mathscr{H}_{i35}&\mathscr{H}_{i36}&\mathscr{H}_{i37}&\mathscr{H}_{i38}\\ \mathscr{H}_{i42}&\mathscr{H}_{i43}&\mathscr{H}_{i44}&\mathscr{H}_{i45}&\mathscr{H}_{i46}&\mathscr{H}_{i47}&\mathscr{H}_{i48}\\ \mathscr{H}_{i52}&\mathscr{H}_{i53}&\mathscr{H}_{i54}&\mathscr{H}_{i55}&\mathscr{H}_{i56}&\mathscr{H}_{i57}&\mathscr{H}_{i58}\\ \mathscr{H}_{i62}&\mathscr{H}_{i63}&\mathscr{H}_{i64}&\mathscr{H}_{i65}&\mathscr{H}_{i66}&\mathscr{H}_{i67}&\mathscr{H}_{i68}\\ \mathscr{H}_{i72}&\mathscr{H}_{i73}&\mathscr{H}_{i74}&\mathscr{H}_{i75}&\mathscr{H}_{i76}&\mathscr{H}_{i77}&\mathscr{H}_{i78}\\ \mathscr{H}_{i82}&\mathscr{H}_{i83}&\mathscr{H}_{i84}&\mathscr{H}_{i85}&\mathscr{H}_{i86}&\mathscr{H}_{i87}&\mathscr{H}_{i88}\end{bmatrix}\begin{bmatrix}\zeta'\\ \zeta''\\ \zeta'''\\ \theta'\\ \theta''\\ \theta'''\\ f\end{bmatrix}$$

Interchangeability of partial differentiation allows some simplification:

$$\mathscr{H}_{ij}=\mathscr{H}_{ji},\qquad \mathscr{H}_{ijk}=\mathscr{H}_{jki}=\mathscr{H}_{kij},\ \ldots$$

since at most three indices appear. For example, the matrix defining $N^{(i)}$ is a symmetric matrix.

The Euler–Lagrange equations are coupled through the adjoint variable $\chi(\lambda)$ because partial derivatives with respect to λ (terms with subscript 1 appearing at least once) introduce total derivative terms χ' and χ'':

$$\mathscr{H}_{ij1}=\mathscr{L}_{ij1}+\chi f_{ij1}+\chi' f_{ij}$$

$$\mathscr{H}_{i11}=\mathscr{L}_{i11}+\chi f_{i11}+2\chi' f_{i1}+\chi'' f_i$$

since χ is regarded as a dependent variable. The terms χ and χ'' may be eliminated by differentiation and substitution of the adjoint equation. Adjoint variable derivatives are absent from $N^{(i)}$.

Autonomous Case

$\mathscr{L}$ and f are independent of λ:

$$\mathscr{L}_1 = 0, \qquad f_1 = 0, \qquad \mathscr{H}_1 = \chi' f$$

The Euler–Lagrange equations for the extremal space curve can be simplified (somewhat) using the identities

$$\mathscr{H}_{i1} = \chi' f_i, \qquad \mathscr{H}_{ij1} = \chi' f_{ij}, \qquad \mathscr{H}_{i11} = \chi'' f_i$$

The term $N^{(i)}$ is unchanged, since it is independent of partial derivatives with respect to λ.

The autonomous problem admits a first integral of motion that differs from the Hamiltonian function. Derive the integral by taking the total derivative of the Hamiltonian function using the chain rule:

$$\frac{\mathrm{d}\mathscr{H}}{\mathrm{d}\lambda} = \mathscr{H}_1 + \mathscr{H}_2\zeta' + \mathscr{H}_3\zeta'' + \mathscr{H}_4\zeta''' + \mathscr{H}_5\theta' + \mathscr{H}_6\theta'' + \mathscr{H}_7\theta''' + \mathscr{H}_8 f$$

The adjoint equation requires

$$\mathscr{H}_1 + \mathscr{H}_8 f = (\chi' + \mathscr{H}_x) f = 0$$

The remaining terms may be expressed a total derivative when the Euler–Lagrange equations are used to eliminate $\mathscr{H}_2$ and $\mathscr{H}_5$:

$$\mathscr{H}_2\zeta' + \mathscr{H}_3\zeta'' + \mathscr{H}_4\zeta''' = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\zeta'\left(\mathscr{H}_3 - \frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathscr{H}_4)\right) + \zeta''\mathscr{H}_4\right]$$

$$\mathscr{H}_5\zeta' + \mathscr{H}_6\zeta'' + \mathscr{H}_7\zeta''' = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\theta'\left(\mathscr{H}_6 - \frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathscr{H}_7)\right) + \theta''\mathscr{H}_7\right]$$

Collecting terms, it follows that the first integral is

$$\mathscr{F} = \zeta'\left(\mathscr{H}_3 - \frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathscr{H}_4)\right) + \zeta''\mathscr{H}_4 + \theta'\left(\mathscr{H}_6 - \frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathscr{H}_7)\right) + \theta''\mathscr{H}_7 - \mathscr{H}$$

since $\mathrm{d}\mathscr{F}/\mathrm{d}\lambda = 0$. The first integral is the Hamiltonian ($\mathscr{F} = -\mathscr{H}$) when $\mathscr{H}$ depends only on two variables x and χ, such that $\mathscr{H}_3 = \mathscr{H}_4 = \mathscr{H}_6 = \mathscr{H}_7 = 0$.

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